torus exceeds by practically a factor of two its value for a cylindrical explosion; the experimental results confirm the tendency toward an increase of the pulsation period upon an increase in the radius of the ring; and

d) according to the experimental data, as the radius of the ring decreases (with fulfillment of the condition of maintaining the toroidal nature of the cavity), the fraction of energy necessary for the shock wave increases and amounts to practically 90% for a value $a_0 \approx 150$; as the radius of the ring increases, the energy balance approaches the data for an explosion with cylindrical symmetry.

The results presented for our investigations confirm the practicability of the method proposed in this paper and the pulsation equation (2.5) obtained on this basis for a toroidal cavity in a compressible liquid.

The author is grateful to V. T. Kuzavov for assistance in conducting the experiments.

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# SHOCK STRUCTURE IN A LIQUID CONTAINING GAS BUBBLES WITH NONSTEADY INTERPHASE HEAT TRANSFER

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UDC 532.529.5:533.6.011.72

In this article the one-velocity, two-pressure model of a two-phase mixture [1] is used in conjunction with the heat-conduction equation for the interior of bubbles in a bubble–liquid mixture to describe the structure of a shock wave in such a mixture.

Shock waves in a liquid containing gas bubbles have been investigated theoretically and experimentally [1-4]. The structure of a shock wave in such a medium has been studied with allowance for the compressibility of the host phase as well as two-velocity and two-temperature effects [5], and it has been shown in the same work that in the case of thermal nonequilibrium the role of two-velocity effects becomes inconsequential against the background of the much stronger thermal dissipation. In this connection the present discussion is framed in the one-velocity model for simplification [6]. The objective of the present study is to refine the results of [6] and to test the applicability of the fixed heat-transfer coefficient or Nusselt number determined from the approximation of a thin thermal boundary layer to the case of nonsteady heat transfer between a pulsating bubble and the host liquid.

## §1. Fundamental Equations

We consider the motion of a liquid in which gas bubbles are suspended and for which the following basic assumptions are made [1]: 1) The distances over which the flow parameters experience any appreciable variation are much greater than the distances between bubbles, and the latter distances in turn are much greater than the bubbles themselves (i.e., the contents by volume $\alpha_2$ of the gas phase are small, $\alpha_2 < 0.1$); 2) the mixture is monodisperse, i.e., in every elementary volume all the bubbles are spherical and have the same radius R; 3) viscosity and heat conduction are essential only in interphase processes and, in particular, during bubble pulsations.

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Moscow, Ufa. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 67-74, May-June, 1977. Original article submitted May 13, 1976.

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Moreover, it is assumed that zero mass transfer takes place between phases and the temperature $T_1$ of the liquid (unlike the temperature of the gas in the bubble interior) is constant. The latter condition ($T_1$ = const) is always satisfied for not too high pressures, on account of the predominant mass content of the liquid (which functions as a thermostat), and simplifies the problem greatly because it obviates the need to analyze the energy equation for the liquid.

Calculations have shown [7] that even for very strong bubble compression ($p_e/p_0 \sim 10$), such that the center of the bubble attains high temperatures (of the gas), the temperature $T_\sigma$ of the bubble surface increases only slightly ($T_\sigma \sim 1.1\ T_0$). The pressure in the bubble in this case attains values very much greater than the partial vapor-saturation pressure corresponding to such values of the bubble surface temperature. This fact lends support to the assumption of inconsequential interphase mass transfer.

For the given mixture, working within the notions of continuum theory and following Noordzij [1], we write the differential equations for the conservation of mass of each phase and conservation of momentum of the total mixture in one-dimensional, steady-state motion:

$$
\begin{gathered}
d(\rho_1 v)/dx = 0, \ d(\rho_2 v)/dx = 0, \\
\rho_i = \rho_i^0 \alpha_i, \quad i = 1, 2, \quad \alpha_1 + \alpha_2 = 1, \\
(\rho_1 + \rho_2) v dv/dx = -dp_1/dx,
\end{gathered}
\tag{1.1}
$$

where the subscript i = 1, 2 refers the corresponding parameters to the liquid and gas, respectively; $\alpha_i$, $p_i$, $\rho_i$, and $\rho_i^0$ are the contents by volume, pressure, average density, and true density ~~of~~ the i-th phase; and v is the velocity. We take as the equations of state of the phases

$$
p_2 = (\gamma - 1) c_{V_2} \rho_2^0 T_2, \quad u_2 = c_{V_2} T_2, \quad \rho_1^0 = \text{const}, \tag{1.2}
$$

where $c_{V_2}$, $u_2$, $T_2$, and $\gamma$ are the specific heat at constant volume, specific internal energy, temperature, and adiabatic exponent of the gas.

Instead of the equation used in [6] for the heat input to the second phase we use the heat-conduction equation for the bubble interior:

$$
c_{p_2} \rho_2^0 v \frac{dT_2}{dx} = \frac{\rho_2^0}{\rho_{20}^0 \xi^2} \frac{\partial}{\partial \xi} \left( \lambda_2 \frac{\rho_2^0 y^4}{\rho_{20}^0 \xi^2} \frac{\partial T_2}{\partial \xi} \right) + v \frac{dp_2}{dx}, \tag{1.3}
$$

where $c_{p_2}$ is the specific heat of the gas at constant pressure; y is the spherical Euler coordinate, $0 \le y \le R(t)$; $\lambda_2$ is the thermal conductivity of the gas; and $\xi$ is the Lagrangian coordinate, $0 \le \xi \le R_0$. The subscript 0 refers to the equilibrium state ahead of the wave. For small volume contents of the gas ($\alpha_2 < 0.1$) and not very strong shocks ($p_e/p_0 < 10$), as shown in [7], the boundary condition on the bubble surface can be stated in the form $T_2(R, t) = T_0$, since the liquid has a much greater thermal conductivity and a much smaller thermal diffusivity than the gas.

The equation of continuity for the gas in Lagrangian coordinates is

$$
\frac{\partial y}{\partial \xi} = \frac{\rho_{20}^0 \xi^2}{\rho_2^0 y^2}. \tag{1.4}
$$

The pressure in the bubble is assumed to be homogeneous (homobaricity condition [7]); this condition is guaranteed when the radial velocity of the bubble walls is well below the velocity of sound in the gas.

It will be helpful in what follows to use the pressure differential equation obtained as the integral of Eq. (1.3) subject to the above-stated assumptions and boundary conditions:

$$
v dp_2/dx = -[3(\gamma - 1)/R] q_R - (3\gamma p_2/R) v dR/dx, \tag{1.5}
$$

where $q_R$ is the heat flux from the bubble into the liquid.

The gas pressures and radii of the bubbles must be related by a deformation compatibility condition. Such a condition in the given case is the Rayleigh equation for pulsations of a single spherical bubble in an unbounded incompressible liquid. For the case in question it has the form

$$
\begin{gathered}
R v dw/dx + 3w^2/2 + 4\nu_1 w/R = (p_2 - p_1 - 2\sigma/R)/\rho_1^0, \\
v dR/dx = w,
\end{gathered}
\tag{1.6}
$$

where w is the radial velocity of the bubble wall and $\nu_1$, $\sigma$ are the viscosity coefficient of the liquid and the coefficient of surface tension.

The system of equations is closed. We transform to dimensionless variables and parameters:

$$\begin{gathered} P_i = p_i/p_{20}, \quad V = v/a_*, \quad W = w/a_*, \quad a_*^2 = p_{20}/\rho_{10}, \\ X = x/R_0, \ \eta = \xi/R_0, \ r = R/R_0, \ \delta = y/R_0, \\ \Theta_i = T_i/T_0, \ \ S = 2\sigma/R_0 p_{20}, \ \ \varkappa = \nu_1/R_0 a_*, \\ M_2 = \rho_2/\rho_1, \quad M_{20} = \rho_{20}^0 \alpha_{20}/\rho_{10}^0 \alpha_{10}, \quad z_i = \rho_i^0/\rho_{i0}^0. \end{gathered} \tag{1.7}$$

The system has first integrals deduced from (1.1):

$$\begin{gathered} \alpha_1 V = \alpha_{10} V_0, \ z_2 \alpha_2 V = \alpha_{20} V_0, \\ \alpha_{10} V_0 V (1 + M_{20}) + P_1 = \alpha_{10} V_0^2 (1 + M_{20}) + P_{10}. \end{gathered} \tag{1.8}$$

It is essential to note that a bubble structure exists for $\alpha_2 \leqslant 0.1$, while at moderate pressures ($p \sim 10$ to 30 bars) the ratio of the true densities of the phases $\langle \rho_2^0 \rangle / \rho_1^0 \ll 1$ (for $p = 1$ bar, the ratio $\langle \rho_2^0 \rangle / \rho_1^0 \sim 10^{-3}$). In this case the mass content of gas can be neglected in comparison with unity, because

$$M_{20} = (\rho_{20}^0 / \rho_{10}^0)\, 0(\alpha_2) \ll 0(\alpha_2) \ll 1. \tag{1.9}$$

Using (1.7) and (1.9), we obtain from (1.3)-(1.6) a system of equations in the dimensionless variables

$$dr/dX = W/V, \ \partial\delta/\partial\eta = (\eta/\delta)^2/z_2; \tag{1.10}$$

$$dW/dX = (P_2 - P_1 - 1{,}5W^2 - S/r - 4\varkappa W/r)/rV; \tag{1.11}$$

$$dP_2/dX = -(3(\gamma - 1)/P_{20} a_* rV) q_R - (3\gamma P_2/r) dr/dx; \tag{1.12}$$

$$\frac{d\Theta_2}{dX} = \frac{D_2 r^3 z_2}{R_0 a_* V \eta^2} \frac{\partial}{\partial\eta}\left(\frac{z_2 \delta^4}{\eta^2} \frac{\partial\Theta_2}{\partial\eta}\right) + \frac{\gamma - 1}{\gamma} r^3 \frac{dP_2}{dX}; \tag{1.13}$$

$$\Theta_2(r, X) = 1. \tag{1.14}$$

The remaining variables not involved in the derivative sign are given by finite relations deduced from (1.2) and (1.8):

$$\begin{gathered} P_2 = z_2 \Theta_2, \ V = V_0(\alpha_{10} + \alpha_{20}/z_2), \\ \alpha_2 = \alpha_{20}/(\alpha_{10} z_2 + \alpha_{20}), \ P_1 = P_{10} - \alpha_{10} V_0 (V - V_0). \end{gathered}$$

Next we consider the structure of a plane stationary shock wave, in which the medium goes from an initial equilibrium state (for which a subscript 0 is attached to the corresponding parameters)

$$V = V_0, \ W_0 = 0, \ \Theta_{20} = \Theta_{10} = 1, \ P_{20} = P_{10} + S = 1$$

to a new equilibrium state (indicated by subscript e)

$$V = V_e, \ W_e = 0, \ \Theta_{2e} = \Theta_{1e} = 1, \ P_{2e} = P_{1e} + S/r_e. \tag{1.15}$$

The values of the parameters in state e are determined from finite relations according to the specified initial-state parameters:

$$\alpha_{20}, \ V_0, \ M_{20}, \ P_{10}, \ z_{10} = z_{20} = r_0 = \Theta_0 = 1.$$

On the basis of (1.15) the required relations assume the form

$$\begin{gathered} \alpha_{1e} V_e = \alpha_{10} V_0, \ \alpha_{2e} V_e z_{2e} = \alpha_{20} V_0, \ \alpha_{1e} + \alpha_{2e} = 1, \\ z_{2e} = P_{2e} = r_e^{-3}, \ \alpha_{10} V_0 (V_0 - V_e) = P_{1e} - P_{10}. \end{gathered}$$

We consider the case of small influence due to capillary effects ($S \ll 1$) an assumption that is fully justified for not too small bubbles ($R_0 \sim 1$ mm) in application to the experiments of [2, 3]. Then from the foregoing relations we obtain

$$P_e = P_{1e} = P_{2e} = \alpha_{10} \alpha_{20} V_0^2.$$

## § 2. Calculation of the Shock Structure

To analyze the asymptotic behavior of the system in the vicinity of the initial equilibrium state we linearize the system with respect to the values of the parameters at the point O and seek a solution in the form of an exponential function decaying as $X \to -\infty$ (the spatial coordinate of the point O is $X = -\infty$, and the coordinate of the point e is $X = +\infty$):

$$V = V_0 + A_V \exp(hX),\ r = 1 + A_r \exp(hX),$$
$$W = A_W \exp(hX),\quad P_i = 1 + A_{P_i} \exp(hX), \tag{2.1}$$
$$z_2 = 1 + A_{z_2} \exp(hX),\quad \Theta_2 = 1 + A_{\Theta_2} \exp(hX) \qquad (i = 1, 2), \qquad (\operatorname{Re} h > 0).$$

After linearization the system of fundamental equations is written

$$hA_r = A_W/V_0; \tag{2.2}$$
$$hA_W = (A_{P_2} + \alpha_{10}V_0A_V - 4\varkappa A_W)/V_0; \tag{2.3}$$
$$hA_{P_2} = 3\gamma\left[2\left(\frac{\partial\Theta}{\partial\eta}\right)_{\eta=1}\Big/V_0\,\mathrm{Pe} - hA_r\right]; \tag{2.4}$$
$$hA_{\Theta_2} = \frac{\gamma-1}{\gamma}hA_{P_2} + 2\nabla^2 A_{\Theta_2}/V_0\,\mathrm{Pe}; \tag{2.5}$$
$$A_{P_2} = A_{z_2} + A_{\Theta_2}; \tag{2.6}$$
$$A_V = -\alpha_{20}V_0A_{z_2},\quad A_{P_1} = -\alpha_{10}V_0A_V, \tag{2.7}$$

where $\nabla^2\Theta = \Theta_{\eta\eta} + 2\Theta_\eta/\eta$ and $\mathrm{Pe} = 2R_0a_*/D_2$ is the Peclet number.

The solution of (2.5) satisfying the boundary condition (1.14) and the condition of finite temperature at the center of the bubble has the form

$$A_{\Theta_2} = A\,\mathrm{sh}\left(\eta G^{1/2}\right)/\eta + [3(\gamma-1)/G](AB - GA_r), \tag{2.8}$$

where

$$G = hV_0\mathrm{Pe}/2;\ B = G^{1/2}\,\mathrm{ch}\,G^{1/2} - \mathrm{sh}\,G^{1/2}; \tag{2.9}$$
$$A = 3(\gamma-1)A_r/[\mathrm{sh}\,G^{1/2} + 3(\gamma-1)B/G].$$

We have thus obtained a single-parameter family (the perturbation amplitudes of all the parameters can be expressed in terms of one of the amplitudes). The condition for the existence of a nontrivial solution yields the transcendental equation

$$V_0^2h^2 = 3P_e - 3\gamma/[1 + 3(\gamma-1)B_1/G] - 4\varkappa V_0h, \tag{2.10}$$

in which $B_1 = G^{1/2}\coth G^{1/2} - 1$; capillary effects are neglected ($S \ll 1$), as is the mass content of the gas phase ($M_{20} \ll 1$) in the derivation of this equation.

For low-viscosity liquids and not too small bubbles ($R_0 \sim 1$ mm), in application to the experiments of [2, 3], $\varkappa \ll 1$. We can therefore drop the last term on the right-hand side of Eq. (2.10). In this case Eq. (2.10) can be rewritten

$$\varphi(\lambda) = \lambda + B\lambda/(\lambda^2 - A) + C(\lambda^{1/2}\coth\lambda^{1/2} - 1) = 0, \tag{2.11}$$

where $\lambda = hV_0\mathrm{Pe}/2$; $A = (3/4)P_e\mathrm{Pe}^2$; $B = (3/4)\gamma\,\mathrm{Pe}^2$; $c = 3(\gamma-1)$. The function $\varphi(\lambda)$ is meromorphic because $\varphi_1(\lambda) = \lambda^{1/2}\coth\lambda^{1/2}$ is an analytic function.

For a solution of the type (2.1) only roots of (2.11) with a positive real part are acceptable. We now prove the existence and uniqueness of a root of (2.11) in the right half-plane.

It is known [8] that for a meromorphic function

$$N - P = \Delta_C \arg\varphi(\lambda)/2\pi, \tag{2.12}$$

where N is the number of zeros and P is the number of poles in the domain bounded by a closed curve C.

We take the contour illustrated in Fig. 1. We compute $\Delta_C \arg\varphi(\lambda)$ for this contour as $\delta \to +\infty$, $\varepsilon \to 0$:

$$\Delta_C \arg\varphi(\lambda) = \Delta_{LN} + \Delta_{NS} + \Delta_{SP} + \Delta_{PL}. \tag{2.13}$$

On the arc LN as $\delta \to +\infty$

$$\varphi(\lambda) = \lambda + 0(\lambda). \tag{2.14}$$

On the arc SP

$$\varphi(\lambda) = (1 - B/A + C/3)\lambda + 0(\lambda). \tag{2.15}$$

From (2.14) and (2.15) we obtain for $1 - B/A + C/3 \neq 0$

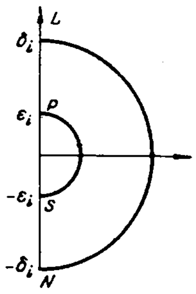


Fig. 1

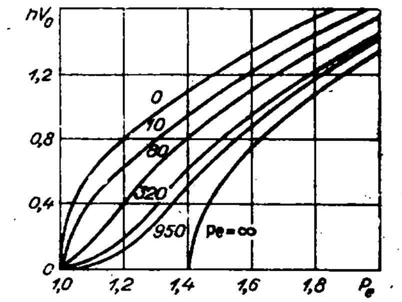


Fig. 2

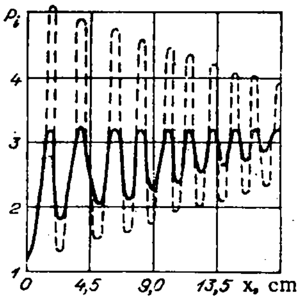


Fig. 3

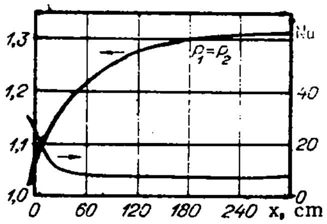


Fig. 4

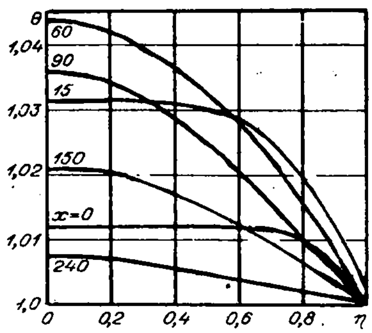


Fig. 5

$$\Delta_{LN} + \Delta_{SP} = 0(1); \tag{2.16}$$

$$\Delta_{NS} \arg \varphi(\lambda) = \Delta_{PL} \arg \varphi(\lambda), \tag{2.17}$$

since $\varphi(\lambda)$ is such that

$$\varphi(\bar{\lambda}) = \overline{\varphi(\lambda)}. \tag{2.18}$$

We infer from (2.12)-(2.18) that

$$N - P = -\Delta_{PL} \arg \varphi(\lambda)/\pi. \tag{2.19}$$

Introducing the parameter $\lambda = iy^2/2$ for the segment OL, we can show that

$$g(y) = (1/C) \operatorname{Re} \varphi(\lambda) = -2 + y(\operatorname{sh} y + \sin y)/(\operatorname{ch} y - \cos y) > 0 \tag{2.20}$$

for $y = 0$. From (2.14), (2.15), (2.19), and (2.20), inserting the values of A, B, and C and acknowledging that $\varphi(\lambda)$ has only one positive $\lambda = +\sqrt{A}$ in the right half-plane, we obtain

For compression waves ($P_e > 1$) the indicated root in the right half-plane exists ($N = 1$). This root, unique in the right half-plane, of the equation $\varphi(\lambda) = 0$ is real, because $\varphi(\lambda)$ has the property (2.18), whence we infer that if $\lambda$ is a root, when $\bar{\lambda}$ is also a root of the equation, $\varphi(\lambda) = 0$. The root of Eq. (2.10) is found with the aid of a computer.

The dependence of the root of the (2.10) on the wave intensity $P_e$ is given in Fig. 2 for various Péclet numbers. In the adiabatic case Eq. (2.10) goes over to the quadratic equation

$$N = 0.5[1 + \operatorname{sgn}(P_e - 1)].$$

from which it is clear that the indicated root exists in the case only for $P_e > \gamma$. With heat transfer present the root exists for all $P_e > 1$.

The integral curves of the system of fundamental equations admit shifting along the X axis. We therefore fix a certain value of the dimensionless bubble radius r at X = 0, choosing r close enough to unity so that a linear solution will hold in the domain $X \leq 0$. We then determine the values of the other parameters at X=0 from (2.2)-(2.9) on the basis of the perturbation amplitude of the bubble radius and the value of the root of Eq. (2.10).

These quantities determine the initial conditions for numerical solution of the nonlinear problem in the domain X > 0. The problem is solved by a finite-difference method in Lagrangian variables; the interior of the bubble is partitioned into spherical layers, and by analogy with [7] a boundary condition at the surface in the form $T_2(R, t) = T_0$ is used. The heat-input equation (1.13) now goes over to a system of n ordinary differential equations (where n is the number of layers), and the continuity equation (1.10) into a system of n algebraic equations. In this way we arrive at the Cauchy problem for the system of (n + 3) ordinary differential equations (1.10)-(1.13). The problem is solved on a computer by the Runge–Kutta method. The number of layers is varied and finally chosen on the basis of the condition that the end results are scarcely affected by increasing that number.

The equilibrium states before and after the shock wave correspond to the points O and e, which are singularities of the system of differential equations. An analysis of the asymptotic behavior as $X \to -\infty$ is necessary in order to cope with the singularities.

We have computed variants of the shock structure in a 1 : 1 glycerin–water solution containing air bubbles in application to the experiments of Noordzij [2] and the corresponding calculations of [6].

The following values are used for the thermodynamic parameters:

$$\rho_{10}^0 = 1126 \text{ kg/m}^3,\ \nu_1 = 75 \cdot 10^{-5} \text{ m}^2/\text{sec},$$
$$T_0 = 300°\text{K},\ c_{V_2} = 716 \text{ m}^2/\text{sec}^2 \cdot °\text{K},$$
$$\lambda_2 = 2.42 \cdot 10^{-2} \text{ kg} \cdot \text{m}/\text{sec}^3 \cdot °\text{K},\ \gamma = 1.4.$$

Figure 3 gives as an example the computed structure of a shock wave with the following values of the parameters determining the initial state of the mixture (wave intensity $P_e = p_e/p_{20}$, wave velocity $v_0$ given relative to the medium ahead of the shock front):

$$R_0 = 1.55 \text{ mm},\ \alpha_{20} = 0.0423,$$
$$p_0 = 0.358 \text{ bar},\quad P_e = 3.32,$$
$$a_* = 5.64 \text{ m/sec},\ V_0 = 9.05\ (v_0 = 50.9 \text{ m/sec});$$

the dashed curve corresponds to the pressure $P_2$ in the bubbles, and the solid curve to the pressure $P_1$ in the liquid.

It has been shown [5] that in strong shocks ($P_e \sim 2$ or 3) each bubble breaks into two identical bubbles at the instant of the first maximum compression of the bubbles. This effect is included in the computations; a discontinuity is introduced at the instant of first maximum compression of a bubble, where the bubble radius is decreased by a factor of $2^{-1/3}$ and the rest of the parameters are left unchanged (the radial velocity of the dividing and already divided bubbles at the instant of breakup is equal to zero). This scheme for taking account of the breakup of bubbles in the wave is greatly simplified. It does not allow for the energy variation in the system during breakup or for energy exchange with the wave. However, it does not contradict the energy balance in the system (for the case in which surface tension can be neglected). Thus, at the instant of maximum compression the energy in the system is the sum of the kinetic energy of macroscopic motion $(\rho_1 + \rho_2)v^2/2$ and the bubble internal energy, which is determined by the pressure in them, while the pulsation energy at this instant is equal to zero. The first two components of the energy (for fixed values of all other parameters) do not depend on the disperseness of the second phase (bubble sizes), and the third component is equal to zero at the instant of breakup.

The shock structure in Fig. 4 is plotted for the following values of the parameters ($Nu = 2Rq_R/[\lambda_2(T_0 - \langle T_2 \rangle)]$ is the dimensionless heat flux, i.e., the Nusselt number):

$$R_0 = 1.4 \text{ mm},\ \alpha_{20} = 0.0246,\ p_0 = 0.902 \text{ bar},$$
$$P_e = 1.32,\quad a_* = 8.95 \text{ m/sec},\ V_0 = 7.42\quad (v_0 = 66.2 \text{ m/sec}).$$

Figure 3 gives an example of a shock wave with a pulsation structure, and Fig. 4 does the same for a monotonic structure. In Fig. 5 we have the temperature distribution inside air bubbles in a weak shock wave at various distances.

The results of the present study demonstrate the applicability of the approximate expressions used in an earlier paper [6] for the interphase heat-transfer coefficient within the context of the two-temperature model. In the case of a shock wave having a pulsation structure (see Fig. 3) the dimensionless heat flux (Nusselt number) also fluctuates, even assuming negative values in certain time intervals (due to the inception of "temperature sinks" in the bubble, as shown by Nigmatulin and Khabeev [7]). However, the period-average value of the Nusselt number and the heat transfer between the bubble and the liquid on the average are well described by the approximate expression of [6]. The radius–time curves calculated by means of Eq. (1.3) and the approximate expressions of [6] practically coincide (they have the same frequency and pulsation decay rate). In the case of a wave having a monotonic structure the value of the Nusselt number initially coincides with the value used in [6] (Nu = 30), but thereafter it exhibits only order-of-magnitude agreement. This fact, however, does not incur any appreciable errors in the results, and it permits considerable simplification of the computations, an asset that is particularly important in the study of nonsteady waves.

The authors are grateful to R. I. Nigmatulin for stating the problem and devoting attention to the work, as well as to A. G. Petrov for a useful discussion.

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